

# On Ramsey properties of classes with forbidden trees

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## Abstract

Let  $\mathcal{T}$  be a set of relational trees and let  $\text{Forb}_h(\mathcal{T})$  be the class of all structures that admit no homomorphism from any tree in  $\mathcal{T}$ ; all this happens over a fixed finite relational signature  $\sigma$ . There is a natural way to expand  $\text{Forb}_h(\mathcal{T})$  by unary relations to an amalgamation class. This expanded class, enhanced with a linear ordering, has the Ramsey property. Both forbidden trees and Ramsey properties have previously been linked to the complexity of constraint satisfaction problems.

**Keywords:** forbidden substructure; amalgamation; Ramsey class; partite method  
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## 1 Introduction

Ramsey's Theorem [22] states the following:

*Given any  $r$ ,  $n$ , and  $\mu$  we can find an  $m_0$  such that, if  $m \geq m_0$  and the  $r$ -element subsets of any  $m$ -element set  $\Gamma$  are divided in any manner into  $\mu$  mutually exclusive classes  $C_i$  ( $i = 1, 2, \dots, \mu$ ), then  $\Gamma$  must contain an  $n$ -element subset  $\Delta$  such that all the  $r$ -element subsets of  $\Delta$  belong to the same  $C_i$ .*

In this paper we study generalizations of Ramsey's Theorem in the context of the so-called *structural Ramsey theory*.

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**Relational structures.** A *signature*  $\sigma$  is a set of relation symbols; each of the symbols has an associated *arity*; the arity of  $R$  is  $\text{ar}(R)$ . A  $\sigma$ -*structure*  $A$  is a set of elements, called the *domain* of  $A$ , together with a relation  $R^A$  on the domain of arity  $\text{ar}(R)$  for every relation symbol  $R \in \sigma$ . An *ordered*  $\sigma$ -*structure* is a  $(\sigma \cup \{\leq\})$ -structure  $A$  such that  $\leq^A$  is a linear ordering. A  $\sigma$ -structure  $A$  is a *substructure* of a  $\sigma$ -structure  $B$  if  $\text{dom } A \subseteq \text{dom } B$  and for each  $k$ -ary  $R \in \sigma$  we have  $R^A = R^B \cap (\text{dom } A)^k$ . We write  $A \subseteq B$  if  $A$  is a substructure of  $B$ .

An *embedding* of  $A$  into  $B$  is a one-to-one mapping  $f : \text{dom } A \rightarrow \text{dom } B$  such that for any  $R \in \sigma$  and any tuple  $\bar{x}$  we have  $\bar{x} \in R^A$  iff  $f(\bar{x}) \in R^B$ , where  $f$  is applied on  $\bar{x}$  component-wise. If  $\sigma \subset \tau$ , the  $\sigma$ -*reduct* of a  $\tau$ -structure  $A$  is the  $\sigma$ -structure  $A^*$  obtained from  $A$  by leaving out all the relations  $R^A$  for  $R \in \tau \setminus \sigma$ . (In some literature a reduct is called a *shadow*.)

**Ramsey classes.** For any structures  $A, B$ , let  $\binom{B}{A}$  denote the set of all embeddings of  $A$  into  $B$ . The partition arrow  $C \rightarrow (B)_r^A$  means that whenever  $\binom{C}{A} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots \cup \mathcal{E}_r$  (a *colouring* with  $r$  colours), then there exists  $g \in \binom{C}{B}$  and  $j \leq r$  such that  $\binom{g[B]}{A} \subseteq \mathcal{E}_j$ . In this case we call  $g$  (or  $g[B]$ ) a *monochromatic copy* of  $B$  in  $C$ .

Let  $\mathcal{C}$  be a class of finite structures and let  $A \in \mathcal{C}$ . The class  $\mathcal{C}$  has the *A-Ramsey property* if for any  $B \in \mathcal{C}$  and any natural number  $r$  there exists  $C \in \mathcal{C}$  such that  $C \rightarrow (B)_r^A$ . The class  $\mathcal{C}$  is called a *Ramsey class* if it has the *A-Ramsey property* for all  $A \in \mathcal{C}$ .

The most notable result about Ramsey classes is most likely the following:

**Theorem 1.1** (Nešetřil–Rödl [15]). *Let  $\sigma$  be a finite relational signature. Then the class of all finite ordered  $\sigma$ -structures is a Ramsey class.*

The presence of orderings is indeed essential; cf. the discussion in [12].

**Classes with forbidden homomorphic images.** Let  $A, B$  be  $\sigma$ -structures. A *homomorphism* of  $A$  to  $B$  is a mapping  $f : \text{dom } A \rightarrow \text{dom } B$  such that for any  $R \in \sigma$  and any  $\bar{x} \in R^A$  we have  $f(\bar{x}) \in R^B$ .

The interest of this paper lies in classes of finite  $\sigma$ -structures that can be defined by forbidding the existence of a homomorphism from a given set of structures. More explicitly, for a set  $\mathcal{F}$  of  $\sigma$ -structures let  $\text{Forb}_h(\mathcal{F})$  be the class of all finite  $\sigma$ -structures  $A$  such that whenever  $F \in \mathcal{F}$ , there exists no homomorphism of  $F$  to  $A$ . We also say that  $A$  is  $\mathcal{F}$ -*free*.

In general, such classes are not Ramsey classes. A Ramsey class of structures always has the *amalgamation property* (see [12]) but these classes will usually not possess it. Following Hubička–Nešetřil [10], however, there is a *canonical way* to add new relations to the signature  $\sigma$  in order to obtain the amalgamation property. Thus it is natural to ask whether this *expanded class*, enhanced with a linear ordering, is a Ramsey class.

**Main result.** It has recently been announced by Nešetřil [14] that the ordered expanded class is a Ramsey class if  $\mathcal{F}$  is a **finite** set of finite connected  $\sigma$ -structures. Here a similar result (Theorem 4.1) is shown for **infinite**  $\mathcal{F}$ , but under the assumption that all elements of  $\mathcal{F}$  are (relational) **trees**. See next section for the definition of a relational tree.

**Complexity of constraint satisfaction.** Constraint satisfaction problems are an important concept in many areas of computer science. Their complexity has been subject to extensive research in the past few years, see [5, 6]. Important tractable cases arise in situations where the class of satisfying instances can be expressed as  $\text{Forb}_h(\mathcal{F})$  for a “simple” class  $\mathcal{F}$ . Perhaps the best-known among them is the case where all structures in  $\mathcal{F}$  are trees, see [9, 8]. Then we speak about *templates with tree duality*. Interestingly, a Ramsey property (which is in fact a special case of the results of this paper) has recently been used to provide a new answer to the classification problem of templates with tree duality, see [1]. Another link between Ramsey theory and CSPs with *infinite templates* has been provided in [2].

**Proof method.** We use the *partite method* of Nešetřil and Rödl [16, 17, 19]. To prove the *partite lemma*, which is often proved by an application of the Hales–Jewett theorem (as in [18, 19, 20]), we apply induction. Our proof is inspired by one of Prömel and Voigt [21].

**Conventions.** 1. A tuple has a bar, so  $\bar{x} = (x_1, x_2, \dots, x_k)$  for some  $k$ . If  $M$  is the domain of some function  $f$  and  $\bar{x} \in M^k$ , then  $f(\bar{x}) = (f(x_1), f(x_2), \dots, f(x_k))$ .

2. Instead of “substructure of  $X$  generated by  $M$ ” I write “substructure of  $X$  induced by  $M$ ” with the intended connotation that the domain of such a substructure is actually  $M$ .

3. For a  $(\sigma \cup \tau)$ -structure  $A$ ,  $A^*$  almost always denotes the  $\sigma$ -reduct of  $A$ .

4. Usually  $R \in \sigma$  and  $S \in \tau$ , but sometimes  $R \in \sigma \cup \tau$ .

## 2 Amalgamation and other constructions

**Amalgamation.** A class  $\mathcal{C}$  of finite structures has the *joint-embedding property* if for any structures  $A_1, A_2 \in \mathcal{C}$  there exists  $B \in \mathcal{C}$  such that both  $A_1$  and  $A_2$  admit an embedding into  $B$ . A class  $\mathcal{C}$  of finite structures has the *amalgamation property* if for any  $A, B_1, B_2 \in \mathcal{C}$  and any embeddings  $f_1 : A \rightarrow B_1$  and  $f_2 : A \rightarrow B_2$  there exists  $C \in \mathcal{C}$  and embeddings  $g_1 : B_1 \rightarrow C$  and  $g_2 : B_2 \rightarrow C$  such that  $g_1 f_1 = g_2 f_2$ . The amalgamation is *free* if  $\text{dom } C = g_1[\text{dom } B_1] \cup g_2[\text{dom } B_2]$  and  $R^C = g_1[R^{B_1}] \cup g_2[R^{B_2}]$  for all  $R \in \sigma$ . If the latter is true only for  $R \in \tau \subset \sigma$ , the amalgamation is said to be *free with respect to  $\tau$* .

Let  $\mathcal{F}$  be a possibly infinite set of finite connected  $\sigma$ -structures. The class  $\text{Forb}_h(\mathcal{F})$  is hereditary and closed under taking disjoint unions, hence it has the joint embedding property. We turn it into an amalgamation class by adding new relations.

**Sum.** For two  $\sigma$ -structures  $A, B$ , their *sum*  $A + B$  is defined by

$$\begin{aligned} \text{dom}(A + B) &= (\{A\} \times \text{dom } A) \cup (\{B\} \times \text{dom } B), \\ R^{A+B} &= (\{A\} \otimes R^A) \cup (\{B\} \otimes R^B), \end{aligned}$$

where

$$\{X\} \otimes R^X = \{((X, x_1), (X, x_2), \dots, (X, x_k)) : (x_1, x_2, \dots, x_k) \in R^X\}.$$

The definition can be extended to arbitrary finite sums in the obvious way. We may also write  $\coprod\{A_1, A_2, \dots, A_k\}$  for  $A_1 + A_2 + \dots + A_k$ . If all elements of  $\mathcal{F}$  are connected, as we assume throughout this paper, then both  $\text{Forb}_h(\mathcal{F})$  and the expanded class  $\mathcal{C}$  are closed under taking sums.

**Factor structure.** If  $A$  is a  $\sigma$ -structure and  $\sim$  is an equivalence relation on  $\text{dom } A$ , let the *factor structure*  $A/\sim$  be defined on  $\text{dom } A/\sim = (\text{dom } A)/\sim$  (the set of all equivalence classes of  $\sim$ ) by letting  $(X_1, X_2, \dots, X_k) \in R^{A/\sim}$  if and only if there exist  $x_1 \in X_1, x_2 \in X_2, \dots, x_k \in X_k$  such that  $(x_1, x_2, \dots, x_k) \in R^A$ .

**Join.** Let  $A, B$  be  $\sigma$ -structures, let  $a_1, a_2, \dots, a_k \in \text{dom } A$  and  $b_1, b_2, \dots, b_k \in \text{dom } B$ . The *join*  $(A, (a_1, a_2, \dots, a_k)) \oplus (B, (b_1, b_2, \dots, b_k))$  is the factor structure  $(A + B)/\sim$ , where  $\sim$  is the minimal equivalence relation such that  $(A, a_i) \sim (B, b_i)$  for each  $i = 1, 2, \dots, k$ . In other words, it is obtained from the disjoint union of  $A$  and  $B$  by identifying every pair  $a_i, b_i$ . In the obvious way, this definition can be extended to joins of more than two structures, too.

**Incidence graph.** The *incidence graph*  $\text{Inc}(X)$  of a  $\sigma$ -structure  $X$  is the bipartite undirected multigraph whose vertex set is  $\text{dom } X \cup \{R^X \times \{R\} : R \in \sigma\}$ , and which contains for every  $R \in \sigma$ , every  $\bar{x} \in R^X$ , and every  $i$ , an edge joining  $(\bar{x}, R)$  and  $x_i$ .

A  $\sigma$ -structure  $X$  is *connected* if  $\text{Inc}(X)$  is connected;  $X$  is a *tree* (or a  $\sigma$ -*tree*) if  $\text{Inc}(X)$  is a tree. (Thus in particular  $X$  is *not* a tree if some tuple of some relation of  $X$  contains the same element two or more times.)

**Pieces.** Without loss of generality let us assume that the domain of each  $F \in \mathcal{F}$  is the set  $\{1, 2, \dots, |F|\}$ . A *cut* of some  $F \in \mathcal{F}$  is a set  $C \subset \text{dom } F$  such that  $\text{Inc}(F) \setminus C$  has at least two distinct connected components that contain vertices from  $\text{dom } F$ ; a *minimal cut* is a cut which is inclusion-minimal.

Let  $C$  be any minimal cut of  $F$  and let  $D$  be the vertex set of some connected component of  $\text{Inc}(F) \setminus C$  that contains a vertex from  $\text{dom } F$ . A *piece* of  $F$  is  $(M, (m_1, \dots, m_k))$ , where  $M$  is the substructure of  $F$  induced by  $C \cup (D \cap \text{dom } F)$  and  $\{m_1, \dots, m_k\} = C$  so that  $m_1 < m_2 < \dots < m_k$ .

*Remarks.* 1.  $C$  is a (minimal) cut of a structure if and only if it is a (minimal) vertex cut of its Gaifman graph.

2.  $\{m_1, \dots, m_k\} = C$  is the set of all elements of  $M$  appearing in some tuple of  $F$  that is not a tuple of  $M$ .

3. A piece of  $F$  is a nonempty connected substructure of  $F$ ,  $M \neq F$ , and  $C \neq \text{dom } M$ .

4. For any given minimal cut, the corresponding pieces cover  $\text{dom } F$ .

**Expansion.** Suppose  $A$  is a  $\sigma$ -structure and  $\{a_1, a_2, \dots, a_k\} \subseteq \text{dom } A$ . Define

$$\begin{aligned} \mathfrak{M}(A, (a_1, a_2, \dots, a_k)) &= \{(M, (m_1, m_2, \dots, m_k)) : \\ &\quad (M, (m_1, m_2, \dots, m_k)) \text{ is a piece of some } F \in \mathcal{F} \text{ and} \\ &\quad (A, (a_1, a_2, \dots, a_k)) \oplus (M, (m_1, m_2, \dots, m_k)) \in \mathcal{F}\}. \end{aligned}$$

Let  $\mathcal{P}(\mathcal{F})$  be the set of all nonempty  $\mathfrak{M}(A, (a_1, a_2, \dots, a_k))$  for all  $\sigma$ -structures  $A$ , all  $k \geq 1$ , and all  $(a_1, a_2, \dots, a_k) \in (\text{dom } A)^k$ .

Let  $\tau$  contain a relation symbol  $S_{\mathfrak{M}}$  for each  $\mathfrak{M} \in \mathcal{P}(\mathcal{F})$ . Let  $\tilde{\mathcal{C}}$  be the class of finite  $(\sigma \cup \tau)$ -structures such that  $A$  belongs to  $\tilde{\mathcal{C}}$  if and only if the  $\sigma$ -reduct  $A^*$  of  $A$  is in  $\text{Forb}_h(\mathcal{F})$  and for any  $\mathfrak{M} \in \mathcal{P}(\mathcal{F})$  and any  $k$ -tuple  $\bar{x} \in (\text{dom } A)^k$  we have

$$\bar{x} \in S_{\mathfrak{M}}^A \iff \exists (M, (m_1, \dots, m_k)) \in \mathfrak{M}, \exists f: M \rightarrow A^* \text{ with } f(m_i) = x_i \text{ for all } i. \quad (2.1)$$

Let  $\mathcal{C}$  be the class of all substructures of the structures in  $\tilde{\mathcal{C}}$ . The class  $\mathcal{C}$  is called the *expanded class* for  $\text{Forb}_h(\mathcal{F})$ . The structures in  $\mathcal{C}$  are called *canonical*. We can also say that  $A$  is  $\mathcal{F}$ -free if  $A^* \in \text{Forb}_h(\mathcal{F})$ ; so being  $\mathcal{F}$ -free is a necessary but not sufficient condition for membership in  $\mathcal{C}$ .

**Theorem 2.1.** *Let  $\sigma$  be a finite relational signature, let  $\mathcal{F}$  be a set of finite connected  $\sigma$ -structures and let  $\mathcal{C}$  be the expanded class for  $\text{Forb}_h(\mathcal{F})$ . Then*

- (1) *the class of all  $\sigma$ -reducts of the structures in  $\mathcal{C}$  is  $\text{Forb}_h(\mathcal{F})$ ;*
- (2)  *$\mathcal{C}$  is closed under isomorphism;*
- (3)  *$\mathcal{C}$  is closed under taking substructures;*
- (4)  *$\mathcal{C}$  has the amalgamation property (free with respect to  $\sigma$ ).*

A similar theorem was proved by Hubička and Nešetřil [10] for finite  $\mathcal{F}$ . Here claims (1), (2) and (3) are obvious from the definition of  $\mathcal{C}$ . Claim (4) can be proved analogously to the finite case. If all structures in  $\mathcal{F}$  are trees, then (4) can be proved by taking the factor structure  $(B_1 + B_2)/\sim$ , where  $\sim$  is the minimal equivalence relation such that  $(B_1, f_1(a)) \sim (B_2, f_2(a))$  for all  $a \in \text{dom } A$ , with the obvious embeddings  $g_1, g_2$ .

*Remarks.* 1. If all structures in  $\mathcal{F}$  are *irreducible*, that is, any two elements lie in a common tuple, then there are no pieces because there are no cuts. Hence the theorem implies that the class  $\text{Forb}_h(\mathcal{F})$  has the amalgamation property (without any new relations).

2. If all structures in  $\mathcal{F}$  are trees, then every minimal cut has size one. Thus all the relations in  $\tau$  are unary. Every piece of a tree is a tree. Moreover,  $\{x\}$  is a minimal cut of  $F$  if and only if  $x$  is an element of  $F$  that belongs to more than one tuple of the relations of  $F$ .

3. Let all elements of  $\mathcal{F}$  be trees. By the results of [7], there exists a finite  $\sigma$ -structure  $H$  such that  $\text{Forb}_h(\mathcal{F}) = \{X : X \rightarrow H\}$  if and only if  $\mathcal{P}(\mathcal{F})$  is finite. This corresponds to *constraint satisfaction problems* with finite templates. In this case, also  $\tau$  is finite and the expanded class  $\mathcal{C}$  has finite signature  $\sigma \cup \tau$ .

4. If all relations in  $\tau$  are unary, then  $\mathcal{C}$  has free amalgamation.

5. Every structure in  $\mathcal{C}$  satisfies the right-to-left implication in (2.1).

6. If  $\mathfrak{M} = (M, (m_1, \dots, m_k))$  is a piece such that there is a homomorphism to  $M$  from some  $F' \in \mathcal{F}$ , then  $S_{\mathfrak{M}}^A = \emptyset$  for any  $A \in \mathcal{C}$ .

**Canonizing.** Suppose  $\mathcal{F}$  is a set of trees, and let  $\mathcal{C}$  be the expanded class for  $\text{Forb}_h(\mathcal{F})$ . Given a  $(\sigma \cup \tau)$ -structure  $A$ , we want to find a superstructure  $\tilde{A}$  of  $A$  that satisfies the left-to-right implication of (2.1). This is possible assuming that

$$\text{every one-element substructure of } A \text{ is in } \mathcal{C}. \quad (2.2)$$

For every  $x \in \text{dom } A$ , let  $A_x$  be the substructure of  $A$  induced by  $\{x\}$ . By assumption, for every  $x$  we have  $A_x \in \mathcal{C}$ ; so there exists  $\tilde{A}_x \in \tilde{\mathcal{C}}$  containing  $A_x$ . Let

$$A' = A + \coprod \{\tilde{A}_x : x \in \text{dom } A\}$$

and let  $\sim$  be the smallest equivalence relation on  $\text{dom } A'$  such that  $(A, x) \sim (\tilde{A}_x, x)$  for all  $x \in \text{dom } A$ . Let  $\tilde{A} = A'/\sim$ .

By convention, we will still use  $x$  to denote the element  $[(A, x)]_\sim$  of  $\tilde{A}$ .

Whenever  $x \in S_{\mathfrak{M}}^{\tilde{A}}$ , then there exist  $(M, (m)) \in \mathfrak{M}$  and  $f : M \rightarrow \tilde{A}_x$  such that  $f(m) = x$ , because  $\tilde{A}_x \in \tilde{\mathcal{C}}$ . Hence  $\tilde{A}$  satisfies the left-to-right implication of (2.1). Moreover, every one-element substructure of  $\tilde{A}$  is isomorphic to a substructure of some  $\tilde{A}_x$ , and so in  $\mathcal{C}$ .

**Proving membership in  $\mathcal{C}$ .** For the following two lemmas, suppose that  $\mathcal{F}$  is a set of finite  $\sigma$ -trees and let  $\mathcal{C}$  be the expanded class for  $\text{Forb}_h(\mathcal{F})$ .

**Lemma 2.2.** *Let  $E = E(F, m)$  be the  $\sigma$ -structure obtained from some  $F \in \mathcal{F}$  with a cut  $m \in \text{dom } F$  so that*

$$\begin{aligned} \text{dom } E &= \{1\}; \\ \text{for each unary } R \in \sigma, \quad 1 \in R^E &\text{ iff } m \in R^F; \\ \text{all the non-unary } \sigma\text{-relations of } E &\text{ are empty}; \\ \text{for each } S_{\mathfrak{M}} \in \tau, \quad 1 \in S^E &\text{ iff } \mathfrak{M} \text{ contains a piece } (M, (m)) \text{ of } F. \end{aligned}$$

*If  $X$  is a  $(\sigma \cup \tau)$ -structure such that there exists a homomorphism  $E \rightarrow X$ , then  $X \notin \mathcal{C}$ .*

*Proof.* Let  $F \in \mathcal{F}$ , let  $m \in \text{dom } F$  be a cut of  $F$ , and consider  $E = E(F, m)$ . Let the corresponding pieces of  $F$  be  $(M_1, (m)), (M_2, (m)), \dots, (M_k, (m))$ .

Let  $X$  be a  $(\sigma \cup \tau)$ -structure. For the sake of contradiction, suppose that there is a homomorphism  $f : E \rightarrow X$  but  $X \notin \mathcal{C}$ . Then there is a canonical superstructure  $\tilde{X} \in \tilde{\mathcal{C}}$  of  $X$ .

Now we recursively construct trees  $F_0, F_1, \dots, F_k \in \mathcal{F}$  in the following way. Let

$$F_0 = (M_1, (m)) \oplus \dots \oplus (M_k, (m)) \cong F.$$

For  $i \geq 1$ , we will have

$$F_i = (N_1, (n_1)) \oplus (N_2, (n_2)) \oplus \dots \oplus (N_i, (n_i)) \oplus (M_{i+1}, (m)) \oplus \dots \oplus (M_k, (m))$$

where  $(N_i, (n_i)) \in \mathfrak{M}(\tilde{M}_i, m)$ , with

$$\tilde{M}_i = (N_1, (n_1)) \oplus \dots \oplus (N_{i-1}, (n_{i-1})) \oplus (M_{i+1}, (m)) \oplus \dots \oplus (M_k, (m)),$$

is selected so that there exists a homomorphism  $g_i : N_i \rightarrow \tilde{X}$ ,  $g_i(n_i) = f(1)$ . We will always find such a piece because  $(M_i, (m)) \in \mathfrak{M}(\tilde{M}_i, m)$ , thus  $f(1) \in S_{\mathfrak{M}(\tilde{M}_i, m)}^{\tilde{X}}$ , and  $\tilde{X}$  satisfies (2.1).

Now we are in trouble:  $F_k \in \mathcal{F}$ , but the homomorphisms  $g_1, \dots, g_k$  induce a homomorphism  $g : F_k \rightarrow \tilde{X}$ . Therefore  $\tilde{X}$  is not  $\mathcal{F}$ -free, a contradiction.  $\square$

A tuple trace of some  $(x_1, x_2, \dots, x_k) \in R^A$  is the structure  $T$  with  $\text{dom } T = \{1, 2, \dots, k\}$ ;  $R^T = \{(1, 2, \dots, k)\}$ ;  $\tilde{R}^T = \{j : x_j \in \tilde{R}^A\}$  for all unary  $\tilde{R} \in \sigma$ ;  $R'^T = \emptyset$  for any other  $R' \in \sigma \setminus \{R\}$ ;  $S^T = \{j : x_j \in S^A\}$  for  $S \in \sigma$ .

**Lemma 2.3.** *Let  $X$  be a  $(\sigma \cup \tau)$ -structure. Then  $X \in \mathcal{C}$  if and only if each one-element substructure of  $X$  belongs to  $\mathcal{C}$ , and for any  $R \in \sigma$  and any  $\bar{x} \in R^X$ , the tuple trace of  $\bar{x}$  belongs to  $\mathcal{C}$ .*

*Proof.* If  $X \in \mathcal{C}$ , then each substructure of  $X$  is in  $\mathcal{C}$  as well. Let  $X \subseteq \tilde{X} \in \tilde{\mathcal{C}}$ . Consider any  $R \in \sigma$  and  $\bar{x} = (x_1, \dots, x_k) \in R^X \subseteq R^{\tilde{X}}$ . Let  $T$  be the tuple trace of  $\bar{x} \in R^X$ . Let  $T'$  be the sum of  $T$  and  $k$  copies of  $\tilde{X}$ ; let  $\sim$  be the smallest equivalence relation that identifies  $j \in \text{dom } T$  with  $x_j$  in the  $j$ th copy of  $\tilde{X}$ . Let  $\tilde{T} = T' / \sim$ . There is an obvious “projection” homomorphism  $p : \tilde{T} \rightarrow \tilde{X}$ ; the image under  $p$  of  $T$  is the substructure of  $\tilde{X}$  induced by  $\bar{x}$ .

If  $F \in \mathcal{F}$  and  $f : F \rightarrow \tilde{T}$ , then  $pf : F \rightarrow \tilde{X}$ , a contradiction. Thus  $\tilde{T}$  is  $\mathcal{F}$ -free. To show that  $\tilde{T}$  satisfies (2.1), first let  $(M, (m)) \in \mathfrak{M} \in \mathcal{P}(\mathcal{F})$  and let  $g : M \rightarrow \tilde{T}$ . Since  $\tilde{X}$  satisfies (2.1) and  $pg : M \rightarrow \tilde{X}$  is a homomorphism, we have  $pg(m) \in S_{\mathfrak{M}}^{\tilde{X}}$ . Hence, by the definition of  $\tilde{T}$ , we have  $g(m) \in S_{\mathfrak{M}}^{\tilde{T}}$ . Conversely, if  $x \in \text{dom } \tilde{T}$  satisfies  $x \in S_{\mathfrak{M}}^{\tilde{T}}$  for some  $\mathfrak{M} \in \mathcal{P}(\mathcal{F})$ , then  $p(x) \in S_{\mathfrak{M}}^{\tilde{X}}$ , thus there exist  $(M, (m)) \in \mathfrak{M}$  and a homomorphism  $h : M \rightarrow \tilde{X}$  such that  $h(m) = p(x)$ . Mapping every element  $a$  of  $M$  to the element corresponding to  $h(a)$  in the copy of  $\tilde{X}$  within  $\tilde{T}$  that contains  $x$  provides a homomorphism from  $M$  to  $\tilde{T}$  that takes  $m$  to  $x$ . Therefore not only  $\tilde{T}$  is  $\mathcal{F}$ -free but it satisfies (2.1) as well, so  $\tilde{T} \in \tilde{\mathcal{C}}$ . The tuple trace  $T$ , which is a substructure of  $\tilde{T}$ , then belongs to  $\mathcal{C}$ .

The converse implication: Suppose that  $X$  satisfies (2.2) and its tuple traces belong to  $\mathcal{C}$ . Apply the canonizing procedure on  $X$  to get  $\tilde{X}$ . We have observed that  $\tilde{X}$  satisfies the left-to-right implication of (2.1). Now we shall show that it also satisfies the right-to-left implication.

Let  $\tilde{X}^*$  be the  $\sigma$ -reduct of  $\tilde{X}$ , let  $\mathfrak{M} \in \mathcal{P}(\mathcal{F})$  and let  $(M, (m)) \in \mathfrak{M}$  be a piece of some  $F \in \mathcal{F}$  and consider any homomorphism  $f : M \rightarrow \tilde{X}^*$  such that  $f(m) \in \text{dom } X$ . We want to show that  $f(m) \in S_{\mathfrak{M}}^{\tilde{X}}$ . For the sake of contradiction, assume that  $f(m) \notin S_{\mathfrak{M}}^{\tilde{X}}$  and that  $M$  is a minimal such piece, that is, we assume that whenever  $N \subset M$  and  $(N, (n))$  is a piece of  $F$ ,  $(N, (n)) \in \mathfrak{N} \in \mathcal{P}(\mathcal{F})$ , then  $f'(n) \in S_{\mathfrak{N}}^{\tilde{X}}$  for any homomorphism  $f' : N \rightarrow \tilde{X}^*$ .

If the piece  $M$  consists of a unique tuple  $\bar{x} \in R^M$ , then the tuple trace of  $f(\bar{x}) \in R^X$  is not in  $\mathcal{C}$ , a contradiction. Hence  $M$  has more than one tuple. Because  $\{m\}$  is a cut of the tree  $F$ ,  $m$  belongs to a unique tuple  $\bar{x}$  of  $M$ ,  $\bar{x} \in R^M$  for some  $R \in \sigma$ ;  $m = x_j$ ;  $f(\bar{x}) \in R^{\tilde{X}}$ . As  $M$  has more than one tuple,  $\bar{x}$  contains at least one element  $n \neq m$  such that  $\{n\}$  is a minimal cut of  $F$ . Let  $(N_1, (n_1))$ ,  $(N_2, (n_2))$ ,  $\dots$ ,  $(N_\ell, (n_\ell))$  be all the pieces of  $F$  corresponding to all minimal cuts  $\{n_k\}$  such that  $n_k = x_i$  for some  $i \neq j$ , and  $m \notin \text{dom } N_k$ . Notice that each  $N_k \subset M$ ; thus by minimality of the counterexample, for each  $k = 1, \dots, \ell$  and any  $\mathfrak{N} \in \mathcal{P}(\mathcal{F})$  such that  $n_k \in \mathfrak{N}$  we have  $f(n_k) \in S_{\mathfrak{N}}^{\tilde{X}}$ . But then the tuple trace of  $f(\bar{x}) \in R^X$  is not in  $\mathcal{C}$ , a contradiction.

Next we show that  $\tilde{X}$  is  $\mathcal{F}$ -free. Suppose there is some  $F \in \mathcal{F}$  and a homomorphism  $f : F \rightarrow \tilde{X}^*$ . Then the image of  $f$  contains elements of  $X$ . If  $F$  has only one element, then the one-element substructure  $f[F]$  of  $X$  is not in  $\mathcal{C}$ . If  $F$  has more than one element but it is irreducible (that is, if it contains exactly one tuple of a relation of arity more than one), then the tuple trace of  $f[F]$  is not in  $\mathcal{C}$ , a contradiction. Hence there is a cut  $\{m\}$  of  $F$  such that  $f(m) \in \text{dom } X$ . Also,

for any piece  $(N, (m)) \in \mathfrak{N}$  of  $F$  the restriction  $g = f \upharpoonright N$  is a homomorphism  $N \rightarrow \tilde{X}^*$  such that  $g(m) = f(m)$ . Thus  $f(m) \in S_{\mathfrak{N}}^X$  for any such piece  $(N, (m))$  and any  $\mathfrak{N} \ni (N, (m))$ . Let  $E = E(F, m)$  be obtained from  $F$  with the cut  $\{m\}$  as in Lemma 2.2. Then the 1-element substructure of  $X$  induced by  $\{f(m)\}$  admits a homomorphism from  $E$ , so by Lemma 2.2 it is not in  $\mathcal{C}$ , again a contradiction. We conclude that  $\tilde{X}^* \in \text{Forb}_h(\mathcal{F})$ .

Therefore  $\tilde{X} \in \tilde{\mathcal{C}}$ , and so  $X \in \mathcal{C}$ .  $\square$

Note that the “tuple trace” is a necessary complication due to the general context of relational structures. If  $\sigma$  were the signature of digraphs (one binary relation), we could simply test all one- and two-element substructures of  $X$ .

### 3 Partite lemma

**Orderings.** An *ordered  $v$ -structure* is a  $(v \cup \{\leq\})$ -structure  $A$  such that the relation  $\leq^A$  is a linear ordering.

**Definition 3.1.** Let  $\sigma$  be a finite relational signature and let  $\mathcal{F}$  be a set of finite connected  $\sigma$ -structures. The *ordered expanded class* for  $\text{Forb}_h(\mathcal{F})$  is the class  $\tilde{\mathcal{C}}$  of ordered  $(\sigma \cup \tau)$ -structures such that  $A \in \tilde{\mathcal{C}}$  if and only if  $\leq^A$  is a linear ordering and the  $(\sigma \cup \tau)$ -reduct of  $A$  is in the expanded class  $\mathcal{C}$  for  $\text{Forb}_h(\mathcal{F})$ .

**Rectified structures.** Let  $A \in \tilde{\mathcal{C}}$ . An  *$A$ -rectified structure* is a pair  $(X, \iota_X)$  such that  $X \in \tilde{\mathcal{C}}$ ,  $\iota_X : \text{dom } X \rightarrow \text{dom } A$ ,  $x \leq^X x'$  implies that  $\iota_X(x) \leq^A \iota_X(x')$ , and for any  $R \in \sigma \cup \tau$  and any  $\bar{x} \in (\text{dom } X)^{\text{ar}(R)}$  we have

$$\bar{x} \in R^X \iff \iota_X \text{ is injective on } \bar{x} \text{ and } \iota_X(\bar{x}) \in R^A. \quad (3.1)$$

Observe that  $X$  is uniquely determined by  $A$ ,  $\text{dom } X$  and  $\iota_X$  via (3.1).

A mapping  $e : \text{dom } X \rightarrow \text{dom } Y$  is an embedding of  $A$ -rectified structure  $(X, \iota_X)$  into  $(Y, \iota_Y)$  if  $e : X \rightarrow Y$  is an embedding of  $(\sigma \cup \tau \cup \{\leq\})$ -structures and  $\iota_X = \iota_Y e$ .

*Note.*  $(A, \text{id}_A)$  is always  $A$ -rectified; and for any  $A$ -rectified  $(X, \iota_X)$ , any mapping  $e : \text{dom } A \rightarrow \text{dom } X$  such that  $\iota_X e = \text{id}_A$  is an embedding of  $A$  into  $X$ , as well as an embedding of  $(A, \text{id}_A)$  into  $(X, \iota_X)$ .

**Lemma 3.2.** Let  $\mathcal{F}$  be a set of finite connected  $\sigma$ -structures and let  $\tilde{\mathcal{C}}$  be the ordered expanded class for  $\text{Forb}_h(\mathcal{F})$ ; let  $A \in \tilde{\mathcal{C}}$ . Let  $(B, \iota_B)$  be  $A$ -rectified,  $r \geq 1$ . Then there exists  $A$ -rectified  $(E, \iota_E)$  such that  $(E, \iota_E) \rightarrow (B, \iota_B)_r^{(A, \text{id}_A)}$ .

*Proof.* By induction on  $|A|$ . If  $|A| = 1$ , take  $E$  to be the sum (disjoint union) of  $r \cdot (|B| - 1) + 1$  copies of  $A$  with an arbitrary linear ordering  $\leq^E$ ;  $\iota_E$  is constant.

If  $|A| \geq 2$ , assume that  $\text{dom } A = \{0, 1, \dots, n\}$ . Let  $A'$  be the substructure of  $A$  induced by the subset  $\{1, \dots, n\}$ ; let  $B'$  be the substructure of  $B$  induced by  $\iota_B^{-1}[\{1, \dots, n\}]$ , and  $\iota_{B'} = \iota_B \upharpoonright \text{dom } B'$ . Then  $(B', \iota_{B'})$  is  $A'$ -rectified. Apply induction to get  $A'$ -rectified  $(E', \iota_{E'})$  such that  $(E', \iota_{E'}) \rightarrow$



$(B', \iota_{B'})_{r^k}^{(A', \iota_{A'})}$ , where  $k = r \cdot (|\iota_B^{-1}(0)| - 1) + 1$ . Assuming that  $\text{dom } E' \cap \{1, 2, \dots, k\} = \emptyset$  let  $\text{dom } E = \text{dom } E' \cup \{1, 2, \dots, k\}$  and define  $\iota_E(x) = 0$  if  $x \in \{1, 2, \dots, k\}$  and  $\iota_E(x) = \iota_{E'}(x)$  otherwise. Let all  $(\sigma \cup \tau)$ -relations of  $E$  be defined by (3.1); let  $\leq^E$  be an extension of  $\leq^{E'}$  that is preserved by  $\iota_E$ . Thus  $E'$  is the substructure of  $(E, \iota_E)$  on  $\iota_E^{-1}[\{1, \dots, n\}]$ .

Next, to prove that  $(E, \iota_E) \rightarrow (B, \iota_B)_r^{(A, \text{id}_A)}$ , consider any  $r$ -colouring  $\chi$  of  $(\frac{(E, \iota_E)}{(A, \text{id}_A)})$ . Define  $\chi' : (\frac{(E', \iota_{E'})}{(A', \text{id}_{A'})}) \rightarrow \{1, \dots, r\}^{\iota_E^{-1}(0)}$  by  $\chi'(e') = (c \mapsto \chi(e' \cup (0 \mapsto c)))$ , that is, the vector of colours of all extensions of  $e' \in (\frac{(E', \iota_{E'})}{(A', \text{id}_{A'})})$  to some  $e \in (\frac{(E, \iota_E)}{(A, \text{id}_A)})$ . By the definition of  $(E', \iota_{E'})$ , there is a monochromatic  $g' \in (\frac{(E', \iota_{E'})}{(B', \iota_{B'})})$ . Hence for any fixed  $c \in \iota_E^{-1}(0)$ , the mapping  $\varphi_c : h' \mapsto \chi((g' h') \cup (0 \mapsto c))$  is constant on  $(\frac{(B', \iota_{B'})}{(A', \text{id}_{A'})})$ . Define  $\psi : \iota_E^{-1}(0) \rightarrow \{1, \dots, r\}$  by setting  $\psi(c)$  to be the constant value of  $\varphi_c$ . Since  $|\iota_E^{-1}(0)| = k > r (|\iota_B^{-1}(0)| - 1)$ , there exists a subset  $M \subseteq \iota_E^{-1}(0)$  with  $|M| = |\iota_B^{-1}(0)|$  such that  $\psi$  is constant on  $M$ . Define  $g \in (\frac{(E, \iota_E)}{(B, \iota_B)})$  to be an extension of  $g'$  by the  $\leq$ -preserving bijection of  $\iota_B^{-1}(0)$  and  $M$ . Then  $g$  is monochromatic.

Finally, to show that  $(E, \iota_E)$  is  $A$ -rectified we need only to check that  $E \in \vec{\mathcal{C}}$ . First, the  $\sigma$ -reduct  $E^*$  of  $E$  is  $\mathcal{F}$ -free, for if there were a homomorphism  $f : F \rightarrow E^*$  of some  $F \in \mathcal{F}$ , then  $\iota_E f$  would be a homomorphism  $F \rightarrow A^*$  – but  $A$  is  $\mathcal{F}$ -free. Moreover, because  $A \in \vec{\mathcal{C}}$ ,  $A$  is a substructure of a canonical  $\tilde{A}$ . Let  $\text{dom } \tilde{E} = \text{dom } E \cup (\text{dom } \tilde{A} \setminus \text{dom } A)$  (assuming  $\text{dom } E$  and  $\text{dom } \tilde{A}$  are disjoint) and let the relations of  $\tilde{E}$  be defined by (3.1), with  $\iota_{\tilde{E}} = \iota_E \cup \text{id}_{\text{dom } \tilde{E} \setminus \text{dom } E}$ . Clearly  $\tilde{E}$  is canonical and  $E$  is a substructure of  $\tilde{E}$ .  $\square$

## 4 Main result

Recall Definition 3.1 of the ordered expanded class for  $\text{Forb}_h(\mathcal{F})$ .

**Theorem 4.1.** *Let  $\sigma$  be a finite relational signature and let  $\mathcal{F}$  be a set of finite  $\sigma$ -trees. Then the ordered expanded class for  $\text{Forb}_h(\mathcal{F})$  has the Ramsey property.*

The remainder of this section is devoted to the proof of this theorem.

**Partite structures.** Let  $P$  be an ordered  $\sigma$ -structure and let  $\vec{\mathcal{C}}$  be the ordered expanded class for  $\text{Forb}_h(\mathcal{F})$ . A  $P$ -partite  $\vec{\mathcal{C}}$ -structure is a pair  $(A, \iota_A)$  where  $A \in \vec{\mathcal{C}}$  and  $\iota_A : \text{dom } A \rightarrow \text{dom } P$  is a homomorphism of the  $(\sigma \cup \{\leq\})$ -reduct  $A^*$  of  $A$  to  $P$  that is injective on any tuple of the relation  $R^A$  for any  $R \in \sigma$ , and such that the restriction of  $\iota_A$  to any one-element substructure of  $A^*$  is an embedding of this one-element  $(\sigma \cup \{\leq\})$ -structure into  $P$ . A  $P$ -partite  $\vec{\mathcal{C}}$ -structure  $(A, \iota_A)$  is *transversal* if  $\iota_A$  is an embedding of  $A^*$  to  $P$ .

A mapping  $e : \text{dom } A \rightarrow \text{dom } B$  is an embedding of a  $P$ -partite  $\vec{\mathcal{C}}$ -structure  $(A, \iota_A)$  into  $(B, \iota_B)$  if  $e : A \rightarrow B$  is an embedding of  $(\sigma \cup \tau \cup \{\leq\})$ -structures and  $\iota_A = \iota_B e$ .

**Lemma 4.2** (“rectification”). *Let  $\vec{\mathcal{C}}$  be the ordered expanded class for  $\text{Forb}_h(\mathcal{F})$ , where  $\mathcal{F}$  is a set of finite  $\sigma$ -trees. Let  $(C, \iota_C)$  be a  $P$ -partite  $\vec{\mathcal{C}}$ -structure for some  $\sigma$ -structure  $P$ . If  $(D, \iota_D)$  is defined*

by setting

$$\begin{aligned}
& \text{dom } D = \text{dom } C, \\
& \iota_D = \iota_C, \\
& S^D = S^C \text{ for } S \in \tau, \\
& \leq^D = \leq^C, \\
& \text{for } R \in \sigma, \bar{x} \in R^D \iff \iota_D \text{ is injective on } \bar{x}, \text{ and} \\
& \quad \exists \bar{y} \in R^C : \iota_C(\bar{y}) = \iota_D(\bar{x}) \text{ and } \forall i, \forall S \in \tau : x_i \in S^D \Leftrightarrow y_i \in S^C,
\end{aligned} \tag{4.1}$$

then  $(D, \iota_D)$  is a  $P$ -partite  $\vec{\mathcal{C}}$ -structure.

*Proof.* It is straightforward that  $\iota_D$  is a homomorphism of the reduct  $D^*$  to  $P$  because  $\iota_C$  is a homomorphism of  $C^*$  to  $P$ . By definition,  $\iota_D$  is injective on any tuple of any  $\sigma$ -relation of  $D$ , and every one-element substructure of  $D$  is isomorphic to the corresponding one-element substructure of  $C$ .

To show that  $D \in \vec{\mathcal{C}}$ , first apply the “only if” direction of Lemma 2.3 to prove that the tuple trace of any  $\bar{y} \in R^D$  is in  $\vec{\mathcal{C}}$ . Then observe that the tuple trace of any  $\bar{y} \in R^D$  is equal to the tuple trace of some  $\bar{x} \in R^C$ . Also, any one-element substructure of  $D$  is isomorphic to some one-element substructure of  $C$ . Finally apply the “if” direction of Lemma 2.3.  $\square$

Observe that the  $P$ -partite  $\mathcal{C}$ -structure  $(D, \iota_D)$  from Lemma 4.2 is *rectified* in the following sense:

$$\begin{aligned}
& \text{For any } R \in \sigma \text{ and any } \bar{y} \in R^D, \text{ if } \bar{x} \text{ is a tuple such that } \iota_D(\bar{x}) = \iota_D(\bar{y}), \\
& \quad \iota_D \text{ is injective on } \bar{x}, \text{ and } y_i \in S^D \Leftrightarrow x_i \in S^D \text{ for any } i \text{ and any } S \in \tau, \text{ then } \bar{x} \in R^D.
\end{aligned} \tag{4.2}$$

Note that if  $(C, \iota_C)$  satisfies (4.2) and  $(D, \iota_D)$  is defined by (4.1), then  $(D, \iota_D) = (C, \iota_C)$ . An important special case: if  $(C, \iota_C)$  is transversal.

**Lemma 4.3.** *Let  $(D, \iota_D)$  be a  $P$ -partite  $\vec{\mathcal{C}}$ -structure satisfying (4.2), and let  $(A, \iota_A)$  be a transversal  $P$ -partite  $\vec{\mathcal{C}}$ -structure. Suppose there is an embedding of  $(A, \iota_A)$  into  $(D, \iota_D)$ . Define*

$$\text{dom } B = \{x \in \text{dom } D : \iota_D(x) \in \iota_A[\text{dom } A] \text{ and for any } S \in \tau : x \in S^D \Leftrightarrow \iota_A^{-1}(\iota_D(x)) \in S^A\} \tag{4.3}$$

*and let  $B$  be the substructure of  $D$  induced by  $\text{dom } B$ . Set  $\iota_B = \iota_A^{-1}(\iota_D \upharpoonright \text{dom } B)$ . Then  $(B, \iota_B)$  is  $A$ -rectified.*

*Proof.* First,  $B \in \vec{\mathcal{C}}$  because it is a substructure of  $D \in \vec{\mathcal{C}}$ . Since  $(D, \iota_D)$  is  $P$ -partite,  $\iota_D$  is injective on any tuple of any relation of  $B$ , and so is  $\iota_B$ . Because there exists an embedding of  $(A, \iota_A)$  into  $(D, \iota_D)$ , it follows from (4.2) that a mapping  $e : \text{dom } A \rightarrow \text{dom } D$  such that  $\iota_A = \iota_D e$  is an embedding of  $(A, \iota_A)$  into  $(D, \iota_D)$  if and only if for any  $a \in \text{dom } A$  and any  $S \in \tau$  we have  $a \in S^A \Leftrightarrow e(a) \in S^D$ . Therefore  $(B, \iota_B)$  satisfies (3.1).  $\square$

**Proof of Theorem 4.1.** Let  $\mathcal{F}$  be a set of finite  $\sigma$ -trees and let  $\mathcal{C}$  be the expanded class and  $\vec{\mathcal{C}}$  the ordered expanded class for  $\text{Forb}_h(\mathcal{F})$ . Consider  $A, B \in \vec{\mathcal{C}}$  and a positive integer  $r$ . We construct  $C \in \vec{\mathcal{C}}$  such that  $C \rightarrow (B)_r^A$ .

Let  $A^*, B^*$  be the  $(\sigma \cup \{\leq\})$ -reducts of  $A, B$ , respectively. By Theorem 1.1 there exists an ordered  $\sigma$ -structure  $P$  such that  $P \rightarrow (B^*)_r^{A^*}$ . Define  $(C_0, \iota_{C_0})$  by

$$\text{dom } C_0 = \binom{P}{B^*} \times \text{dom } B,$$

for any  $k$ -ary  $R \in \sigma \cup \tau$ :

$$R^{C_0} = \left\{ ((f, x_1), (f, x_2), \dots, (f, x_k)) : f \in \binom{P}{B^*} \text{ and } (x_1, x_2, \dots, x_k) \in R^B \right\},$$

$\iota_{C_0} : \text{dom } C_0 \rightarrow \text{dom } P$  is defined by  $\iota_{C_0} : (f, x) \mapsto f(x)$ ,

$\leq^{C_0}$  is any linear ordering that is preserved by  $\iota_{C_0}$ .

Thus  $C_0$  is isomorphic to a sum of structures, and each of the summands is isomorphic to  $B$ . See Figure 1. Observe that  $(C_0, \iota_{C_0})$  is a  $P$ -partite  $\vec{\mathcal{C}}$ -structure.

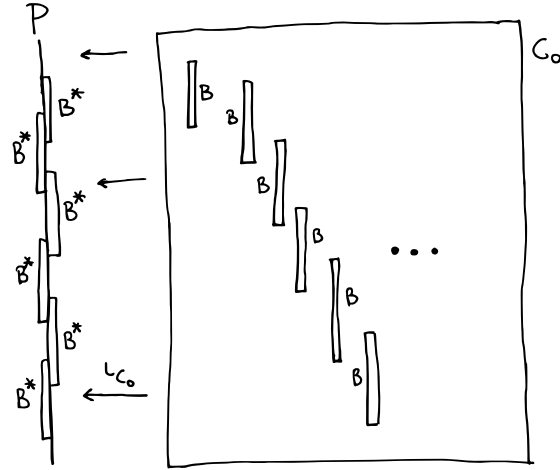


Figure 1:  $C_0$ .

If  $(D_0, \iota_{D_0})$  is obtained from  $(C_0, \iota_{C_0})$  by (4.1), then each of the basic embeddings  $x \mapsto (f, x)$  of  $B$  to  $C_0$  is also an embedding of  $B$  to  $D_0$ .

Fix some numbering of  $\binom{P}{A^*} = \{e_1, \dots, e_N\}$ . We will inductively construct  $P$ -partite  $\vec{\mathcal{C}}$ -structures  $(C_1, \iota_{C_1}), \dots, (C_N, \iota_{C_N})$ .

Let  $k \in \{1, \dots, N\}$  and suppose  $(C_{k-1}, \iota_{C_{k-1}})$  has been constructed. If there is no  $P$ -partite embedding of  $(A, e_k)$  into  $(C_{k-1}, \iota_{C_{k-1}})$ , let  $(C_k, \iota_k) = (C_{k-1}, \iota_{C_{k-1}})$ . Otherwise let  $(D_{k-1}, \iota_{D_{k-1}})$  be defined from  $(C_{k-1}, \iota_{C_{k-1}})$  by (4.1). Let  $(B_k, \iota_{B_k})$  be obtained from  $(D_{k-1}, \iota_{D_{k-1}})$  as in Lemma 4.3, using  $(A, e_k)$  in place of  $(A, \iota_A)$ . Then  $(B_k, \iota_{B_k})$  is  $A$ -rectified and we can apply the Partite Lemma,

Lemma 3.2, in order to get  $A$ -rectified  $(E_k, \iota_{E_k})$  such that  $(E_k, \iota_{E_k}) \rightarrow (B_k, \iota_{B_k})_r^{(A, \text{id}_A)}$  (w.r.t. embeddings of  $A$ -rectified structures). Therefore  $(E_k, e_k \iota_{E_k}) \rightarrow (B_k, e_k \iota_{B_k})_r^{(A, e_k)}$  (w.r.t. embeddings of  $P$ -partite structures). Set

$$\text{dom } C_k = \text{dom } E_k \cup \left( \left( \begin{smallmatrix} (E_k, \iota_{E_k}) \\ (B_k, \iota_{B_k}) \end{smallmatrix} \right) \times (\text{dom } D_{k-1} \setminus \text{dom } B_k) \right).$$

Define  $\lambda_k : \left( \begin{smallmatrix} (E_k, \iota_{E_k}) \\ (B_k, \iota_{B_k}) \end{smallmatrix} \right) \times \text{dom } D_{k-1} \rightarrow \text{dom } C_k$  by

$$\lambda_k : (g, x) \mapsto \begin{cases} g(x) & \text{if } x \in \text{dom } B_k, \\ (g, x) & \text{otherwise.} \end{cases}$$

For any  $\ell$ -ary  $R \in \sigma \cup \tau$ , let

$$R^{C_k} = \left\{ (\lambda_k(g, x_1), \dots, \lambda_k(g, x_\ell)) : g \in \left( \begin{smallmatrix} (E_k, \iota_{E_k}) \\ (B_k, \iota_{B_k}) \end{smallmatrix} \right), (x_1, \dots, x_\ell) \in R^{D_{k-1}} \right\}.$$

Furthermore define  $\iota_{C_k} : \text{dom } C_k \rightarrow \text{dom } P$  by

$$\begin{aligned} \iota_{C_k} : y &\mapsto e_k \iota_{E_k}(y) & \text{if } y \in \text{dom } E_k, \\ \iota_{C_k} : (g, x) &\mapsto \iota_{D_{k-1}}(x) & \text{otherwise.} \end{aligned}$$

Finally, let  $\leq^{C_k}$  be a linear ordering such that  $y \leq^{C_k} y'$  if  $y \leq^{E_k} y'$ ,  $\lambda_k(g, x) \leq^{C_k} \lambda_k(g, x')$  if  $x \leq^{D_{k-1}} x'$ , and  $z \leq^{C_k} z'$  if  $\iota_{C_k}(z) \leq^P \iota_{C_k}(z')$ . See Figure 2.

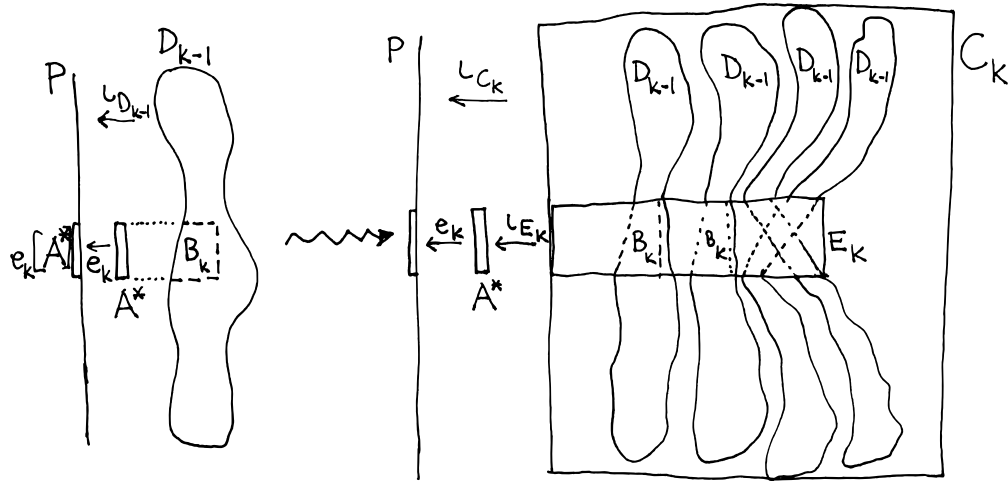


Figure 2:  $C_k$ .

Note that for a fixed  $g$ , the mapping  $\lambda_k(g, -) : x \mapsto \lambda_k(g, x)$  is an embedding of  $(D_{k-1}, \iota_{D_{k-1}})$  to  $(C_k, \iota_{C_k})$ . By definition of  $D_{k-1}$ ,  $\lambda_k(g, -)$  is an injective homomorphism of  $(C_{k-1}, \iota_{C_{k-1}})$  to  $(C_k, \iota_{C_k})$ . The inclusion mapping is an embedding of  $E_k$  to  $C_k$  because  $(E_k, \iota_{E_k})$  is  $A$ -rectified.

Now we claim that  $(C_k, \iota_{C_k})$  is a  $P$ -partite  $\tilde{\mathcal{C}}$ -structure. First, for  $R \in \sigma \cup \tau$ , if  $\bar{x} \in R^{C_k}$ , then  $\bar{x} = \lambda_k(g, \bar{y})$  for some  $(g, \bar{y})$ . Since  $\iota_{D_{k-1}}$  is injective on  $\bar{y}$  and preserves it if  $R \in \sigma$ ,  $\iota_{C_k}$  is injective on  $\bar{x}$  and preserves it if  $R \in \sigma$ . Next,  $\leq^{C_k}$  is preserved by  $\iota_{C_k}$  by definition. The tuple trace of any tuple of any relation of  $C_k$  is the tuple trace of some tuple of the corresponding relation of  $D_{k-1}$ , hence in  $\tilde{\mathcal{C}}$ . By Lemma 2.3,  $C_k \in \tilde{\mathcal{C}}$ .

Let  $C = C_N$ . We show that  $C \rightarrow (B)_r^A$ . Consider any colouring  $\chi : \binom{C}{A} \rightarrow \{1, \dots, r\}$ . By downward induction we exhibit injective homomorphisms  $h_i : (C_{i-1}, \iota_{C_{i-1}}) \rightarrow (C_i, \iota_{C_i})$  for  $i = N, N-1, \dots, 1$  that have certain monochromatic properties.

Suppose  $h_i$  is known for  $i = N, \dots, k+1$  (possibly for no  $i$  yet). If  $(C_k, \iota_{C_k}) = (C_{k-1}, \iota_{C_{k-1}})$ , let  $h_k$  be the identity mapping. Otherwise define the colouring  $\chi_k : \binom{(E_k, \iota_{E_k})}{(A, \text{id}_A)} \rightarrow \{1, \dots, r\}$  by setting  $\chi_k(q) = \chi(h_N h_{N-1} \dots h_{k+1} q)$ . (Observe that the composed mapping is indeed an embedding.) Since  $(E_k, \iota_{E_k}) \rightarrow (B_k, \iota_{B_k})_r^{(A, \text{id}_A)}$ , there exists a  $\chi_k$ -monochromatic embedding  $g_k : (B_k, \iota_{B_k}) \rightarrow (E_k, \iota_{E_k})$ . Let  $h_k = \lambda(g_k, -)$ .

Let  $h = h_N h_{N-1} \dots h_1 : (C_0, \iota_{C_0}) \rightarrow (C_N, \iota_{C_N})$ . Consider any  $e_j \in \binom{P}{A^*}$ . Any embedding  $d$  of  $A$  to  $C_0$  such that  $\iota_{C_0} d = e_j$  is also a  $P$ -partite embedding of  $(A, e_j)$  to  $(C_0, \iota_{C_0})$ . Moreover,  $hd$  is a  $P$ -partite embedding of  $(A, e_j)$  to  $(C_N, \iota_{C_N})$ . By definition of  $h_j$ , all such embeddings take the same colour under  $\chi$ . Thus we define  $\chi_0 : \binom{P}{A^*} \rightarrow \{1, \dots, r\}$  by  $\chi_0(e_j) = \chi(JD)$  if there exists  $d \in \binom{C_0}{A}$  such that  $\iota_{C_0} d = e_j$ , and arbitrarily otherwise. By definition of  $P$  there exists  $\chi_0$ -monochromatic  $f \in \binom{P}{B^*}$ . Let  $c : B \rightarrow C_0$  be the embedding given by  $c : x \mapsto (f, x)$ .

Conclude the proof by observing that  $hc$  is a  $\chi$ -monochromatic embedding of  $B$  to  $C$ : It is an embedding because  $h$  is a composition of embeddings of  $(D_{k-1}, \iota_{D_{k-1}})$  to  $(D_k, \iota_{D_k})$  and the copy of  $B$  given by  $h_k h_{k-1} \dots h_1 c[B]$  remains intact during the “rectification” – application of Lemma 4.2.  $\square$

## 5 Comments

**Universal structures.** If  $\mathcal{F}$  is a set of finite connected  $\sigma$ -structures, then the expanded class for  $\text{Forb}_h(\mathcal{F})$  has a Fraïssé limit  $U$ . The  $\sigma$ -reduct  $U^*$  of  $U$  is a universal structure for  $\text{Forb}_h(\mathcal{F})$ . For finite  $\mathcal{F}$  this universal structure is  $\omega$ -categorical; the existence of such a universal  $\omega$ -categorical structure (and much more) was proved by Cherlin, Shelah and Shi [4]. If  $\mathcal{F}$  is infinite,  $U^*$  is no longer necessarily  $\omega$ -categorical; however, it is model-complete. **(Or is it???)**

**Extreme amenability.** By a theorem of Kechris, Pestov and Todorćević [11], the automorphism group of a Ramsey structure is extremely amenable. Thus Theorem 4.1 provides a continuum of examples of structures with an extremely amenable automorphism group: take  $\mathcal{F}'$  to be an infinite antichain of  $\sigma$ -trees; then the Fraïssé limit of the expanded class for  $\text{Forb}_h(\mathcal{F})$  provides such an example for any subset  $\mathcal{F}$  of  $\mathcal{F}'$ . **(Are their topological automorphism groups all essentially different???)**

**Problem.** It would be interesting to classify all sets  $\mathcal{F}$  of  $\sigma$ -structures for which the corresponding ordered expanded class for  $\text{Forb}_h(\mathcal{F})$  is a Ramsey class. In particular, is it the case for any set  $\mathcal{F}$  of connected finite  $\sigma$ -structures?

**Constraint satisfaction problems.** The problem above is particularly interesting if  $\text{Forb}_h(\mathcal{F})$  defines a  $\text{CSP}(H)$  with a finite template  $H$ , and  $\mathcal{F}$  is some well-behaved complete set of obstructions (e.g., the tree-width of the structures in  $\mathcal{F}$  is bounded by a constant). Some possible applications of such new results are hinted at in [1].

**Limits of the partite method.** Nešetřil [14] asked whether one can prove all Ramsey classes by a variant of the partite (amalgamation) construction. This is certainly a question worth considering. It is not very satisfactory that the definition of a partite structure is rather different each time: compare [3, 13, 16, 17, 18, 19, 20]. Also, the partite lemma is sometimes proved by induction (such as here and in [3, 21]), sometimes by an application of the Hales–Jewett theorem (such as in [18, 19, 20]).

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